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Probability Interpretation of Particle(s) along Classical Trajectories

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Abstract

The theory of divergent series is used to show that the probability density of a wave-mechanical state becomes classical trajectories (for a single particle or many interacting particles) in the limit $\hbar \to 0$. The probability interpretation is claimed to be valid even if a 'classical particle' idea is inserted into the theory. The idea 'particle' is claimed as a classical entity valid for $\hbar \to 0$. Finally, the double slit experiment is illustrated as an example and interpreted by the above theory. The interpretation is that the interaction between the incident and interference beams and the double slit is approximately described by the wave theory (interference mechanism).

1. *Introduction*

The trajectory picture of a particle in quantum mechanics can be used only when the analogy between classical dynamics and geometrical optics is considered. This development can be found systematically in the books of de Broglie (1860), Schrödinger (1929) and Lanczos (1966). As yet, no trajectory picture has been derived from the probability density. On the contrary, the possibility of the probability interpretation almost destroys the trajectory picture. A critical study related to this problem had been surveyed by de Broglie (1964). Two problems (and their related points of view) mentioned in this book will be answered mathematically from quantum mechanics theory in this article. The first problem is that whether the Ψ-function is objective or subjective representation of probabilities of a particle. According to de Broglie's (descriptive) point of view, it is only a subjective one. The second problem is 'how to incorporate a particle in an extensive wave field'. The answers given in this article to these two problems are as follows. First, the Ψ-function is an objective representation of probability; and, secondly, it is meaningless to ask 'how can one incorporate a "classical particle" in Ψ'. The second answer will be expressed more clearly in Section 4. The method I shall use to answer these problems is a rigorous mathematical proof [which was mentioned in an earlier report (Su, 1968)]. The mathematical method used, apart from many other less rigorous ways, is the theory of divergent series (Hardy, 1948; Knopp, 1957). The only result which can answer the above problems (and other related problems) is that the probability density of Ψ-function of a single particle or many particle system, in $\hbar \to 0$, indicates the classical trajectories of particle(s). This classical trajectory, obtained mathematically from the probability interpretation, is by no means trivial. Consequently, in $\hbar \to 0$, the probability interpretation is valid correctly and confirms the classical trajectory of the particle(s). Thus, the 'particle' is concluded to be only a classical entity which appears only in $\hbar \to 0$ in a quantum system.

It is noted that, since the proof in this article is based on the theory of divergent series (rigorously), the theory of divergent series should be basic mathematics for quantum mechanics. If so, we can obtain, in the viewpoint of this author, a consistent theory for quantum mechanics and classical mechanics from the probability interpretation point of view.

2. *A Quantum Mechanical Theorem*

Applying Riesz's process in the theory of divergent series (Hardy, 1948; Knopp, 1957), in the Appendix, we have proved the following:

Theorem

Let H be a Hermitian operator with eigenfunction $\phi_n^{(r)}$ (with possible degeneracy r) corresponding to the eigenvalue E_n. Denoting the subspace spanned by $\phi_n^{(r)}$ for all possible values of r by M, we have a projection operator P_M onto M. Then the limit

$$\begin{aligned}\lim_{\tau\to\infty} \exp\frac{H-E_n}{i\hbar}\tau &= P_M \\ &\equiv \sum_r |\phi_n^{(r)}\rangle\langle\phi_n^{(r)}| \end{aligned} \tag{2.1}$$

Remarks

(i) H may be the Hamiltonian of the system and E_n its eigenvalues.

(ii) The parameter τ may be considered as time if H is time-independent. Then (2.1) is related to the evolution operator for $t \to \infty$.†

For a complete set of commuting observables $A_1,\ldots,A_N$ with eigenvalues $a_1,\ldots,a_N$, (2.1) implies

$$\prod_{i=1}^{N} \lim_{\tau_i\to\infty} \exp\frac{A_i(p,q)-a_i}{i\hbar}\tau_i = |\phi_{a_1,\ldots,a_N}\rangle\langle\phi_{a_1,\ldots,a_N}| \tag{2.2}$$

the pure state projection operator or density operator.

(iii) Since we can formulate the quantum mechanics based on the density operator (Fano, 1957), the left-hand side of (2.1) (after multiplying an equal probability in every degenerate state), or the left-hand side of

† See this journal, p. 233.

(2.2) can be used as density operator to formulate the quantum mechanics as a limit process. Further properties, especially the classical correspondence, are still under investigation by the same author.

3. {q}-*Representation and the Classical Limit*

Since any quantum state is determined up to a phase factor if it is normalized in a certain way, starting with {q}-representation but with an explicit phase, i.e. $[\{\exp[iS(q)/\hbar]\}|q\rangle]$, we can assign the function $S(q)$ uniquely (up to an unimportant difference $2n\pi\hbar$) in the 'energy representation' as follows. The Schrödinger equation in this representation is

$$\left\langle q \middle| \exp\frac{-iS(q)}{\hbar} H(p,q)\phi \right\rangle = E\left\langle q \middle| \exp\frac{-iS(q)}{\hbar}\phi \right\rangle$$

or

$$H\left(-i\hbar\frac{\partial}{\partial q}+\frac{\partial S}{\partial q},q\right)\left\langle q \middle| \exp\frac{-iS}{\hbar}\phi \right\rangle = E\left\langle q \middle| \exp\frac{-iS}{\hbar}\phi \right\rangle$$

where we have used the formula (Dirac, 1958)

$$\exp\frac{-iS}{\hbar}p\exp\frac{iS}{\hbar} = p+\frac{\partial S}{\partial q}$$

In the classical limit $\hbar \to 0$,

$$H\left(\frac{\partial S}{\partial q},q\right) = E$$

is the Hamilton-Jacobi equation. That is, we have assigned $S(q)$ as the Jacobi S function in the phase. For the time-dependent Schrödinger equation, replacing $S(q)$ by

$$W(q,t) = S(q) - Et$$

we have

$$\left\langle q \middle| \exp\frac{-iW}{\hbar} H(p,q)\phi \right\rangle = \left\langle q \middle| \exp\frac{-iW}{\hbar} i\hbar\frac{\partial}{\partial t}\phi \right\rangle$$

The last equation, in the classical limit $\hbar \to 0$, becomes

$$H\left(\frac{\partial W}{\partial q},q\right) = -\frac{\partial W}{\partial t} \tag{3.1}$$

Again, it is the Hamilton–Jacobi equation.

Now applying this representation to the operator in (2.1) with H being the Hamiltonian, and further assuming first that it is non-degenerate with eigenvalue E_n and

$$W(q,t) = S(q) - E_n t$$

Then

$$\langle q'|\phi_n\rangle\langle\phi_n|q\rangle \exp\frac{i(W(q,t)-W(q',t))}{\hbar}$$

$$=\left\langle q'\left|\exp\frac{-iW(q',t)}{\hbar}\left\{\lim_{T\to\infty}\exp\frac{H(p,q)-E_n}{i\hbar}T\right\}\exp\frac{iW(q,t)}{\hbar}\right|q\right\rangle$$

$$=\left\langle q'\left|\exp\frac{i[W(q,t)-W(q',t)]}{\hbar}\left\{\lim_{T\to\infty}\exp\frac{H[p+(\partial W/\partial q),q]+\partial W/\partial t}{i\hbar}T\right\}\right|q\right\rangle$$

In the classical limit $\hbar\to 0$,

$$\langle q'|\phi_n\rangle\langle\phi_n|q\rangle = \phi_n(q')\phi_n^*(q) = \delta\left[H\left(\frac{\partial W}{\partial q},q\right),-\frac{\partial W}{\partial t}\right]\delta(q'-q) \tag{3.2}$$

Therefore the density operator is diagonalized and has only non-vanishing matrix element for those q's (or q''s)

$$H\left(\frac{\partial W}{\partial q},q\right)=-\frac{\partial W}{\partial t} \tag{3.3}$$

From (3.1), the 'probability density' $\phi_n(q')\phi_n^*(q)$ vanishes everywhere except at given point q at the given instant t along the classical trajectory with $E=E_n$. Along this classical trajectory, it has infinite possibility (particle property). In order to find the particle there, the probability interpretation is confirmed to be

$$\int dq'\,\phi_n(q')\phi_n^*(q)=\int dq'\,\delta(q'-q)=1$$

Consequently, as $\hbar$ is small, the solutions both from the probability interpretation of the quantum mechanics and from classical mechanics are the same. Since the limit $\hbar\to 0$ does not create any physical particles, the particle(s) should be there inside the wave-field even if $\hbar$ is finite. That means that the Ψ-function is an objective representation of probabilities.

The classical physical particle represented by the probability density $\delta(q'-q)$ was correct only in the limit $\hbar\to 0$. If the 'particle' is there inside the $\phi^*(q')\phi(q)$ if $\hbar$ is finite, then we do not have a $\delta(q'-q)$ to describe the particle. Thus the 'particle' idea itself is only a classical concept. When we treat the problem in the quantum mechanics, the description of the physical system of 'particle' should be naturally a wave field. This approach might solve the troubles in quantum mechanics and interprets its peculiarities such as probability interpretations, the uncertainty principle, etc.

For the degenerate case, (2.2) is used. Denoting $H = A_1$ and $E_n = a_1$, we have

$$\phi_{a_1,\ldots,a_N}(q')\,\phi^*_{a_1,\ldots,a_N}(q) = \prod_{i=1}^{N} \delta\left[A_i\left(\frac{\partial S}{\partial q}, q\right), a_i\right]\delta(q'-q)$$

in the time-independent $\{\exp(iS/\hbar)|q\rangle\}$-representation above. All classical dynamical quantities of the particle have their own values (equal to the corresponding quantum numbers of the state) along the classical trajectory since along the classical trajectory $p = \partial S/\partial q$ and $A_i(\partial S/\partial q, q) = A_i(p,q) = a_i$.

4. *Interacting Systems*

As it is claimed above that we need not incorporate a particle in an extended wave field specifically we shall discuss, apart from the obvious case in Section 3 above, about more complicated processes, such as collision problems, measurement or disturbance problems, absorption or creation processes, etc. in this section. All the complicated processes indicated above are many interacting particle system. Such a system should have a total Hamiltonian as believed generally by the author. No problems occur in the low-energy physics cases. In the field of high-energy physics, Dirac (1966, 1970) also stressed this concept. Once such a total Hamiltonian can be written down, using the theory given in Sections 2 and 3, the problem is solved. The entire many interacting quantum particle system should be a classical many interacting particle system in the limit $\hbar \to 0$. Every particle is presented there with the probability density $\delta(q'-q)$ for every degree of freedom and moves in its own classical trajectory, as in the classical case. Nothing should be considered such as the case where the incident particle exists in the incident beam, and so on.

Finally, we discuss the double slit interference experiment. If we consider the stationary incident beam and the stationary interference wave-beam and the double slit to form an interacting system, a total Hamiltonian can be written down. The argument above is valid for $\hbar \to 0$, i.e. the interference pattern appears quantum mechanically as a rigorous mathematical result of the mathematics above, and also, for $\hbar \to 0$, in its classical correspondence.

Since such a system is too complicated to be described by a Hamiltonian explicitly, we consider the system approximately as follows. After passing the double slit, the particles must obey the single-particle equations (free-particle equations). In $\hbar \to 0$, the particle appears on the screen individually (de Broglie, 1964) following its own classical trajectory. The interaction of the double slit with the incident and interference beams near the double slit give the initial conditions for each free particle in the interference beam described above. These initial conditions can, approximately or phenomenologically, be described by the wave theory (interference mechanism). Then the interference pattern comes, as it should, from the wave theory.

This kind of approximate argument is similar to that of de Broglie (1964, p. 55).

The classical dynamical idea and the wave-mechanical descriptions are consistent if $\hbar \to 0$ is considered in the classical case. Further, the particle is only a classical entity, i.e. the idea 'particle' is valid only in the limit $\hbar \to 0$.

Appendix

Proof of the Theorem (2.1)

By the Riesz process in the theory of divergent series (Hardy, 1948; Knopp, 1957), if a function $s(t)$ with $s(0) = 0$ is such that

$$\lim_{W\to\infty} \frac{1}{W} \int_0^W s(t)\,dt = s$$

the sequence $\{s(t)\}$ is called 'limitable $R_{\lambda 1}$ with the value s'. Now, specifically, for the function

$$s(t) = \exp(iat) - 1$$

it is easy to show that $\exp(iat) - 1$ is limitable $R_{\lambda 1}$ with the value -1 as $a \neq 0$ and with the value 0 as $a = 0$. This shows that a function can be defined as

$$\delta(x,y) \equiv \lim_{t\to\infty} \exp[i(x-y)t] = \begin{cases} 1 & x = y \\ 0 & x \neq y. \end{cases} \tag{A.1}$$

Less rigorous proofs of (A.1) can be found in Schweber (1961), Goldberger & Watson (1964) or by Fourier transform together with a contour integral.

One property of $\delta(x,y)$ used below is given as

$$\int_{a_1}^{b_1} dx \int_{a_2}^{b_2} dy\, \delta(x,y) f(x,y) = \int_a^b dx f(x,x) \tag{A.2}$$

where $f(x,y)$ is an arbitrary function of x, y and a, b are the smaller ones of a_i's and b_i's, respectively. (A.2) cannot be obtained by the usual integration methods. It is noted that the Dirac δ-function does have the same double integral property as (A.2).

Now, applying (A.1) and (A.2), we shall prove the theorem (2.1) in Section 2. First, for the discrete spectrum, since the Hermitian operator generates a complete basis for the Hilbert space (Halmos, 1957), any arbitrary state ψ can be expanded as

$$\psi = \sum_{m,r} \phi_m^{(r)} \langle \phi_m^{(r)} | \psi \rangle$$

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